HW # 1: 1.20, 1.25, 3.2, 3.9, 3.26, 3.33 - Due Sept. 9, 2011

Solutions

1.20 Find complex numbers $t = z_1 + z_2$ and $s = z_1 - z_2$, both in polar form, for each of the following pairs:

- **a)** $z_1 = 2 + j3$ and $z_2 = 1 - j2$
- **b)** $z_1 = 3$ and $z_2 = -j3$
- **c)** $z_1 = 4\angle 30^\circ$ and $z_2 = 3\angle -30^\circ$
- **d)** $z_1 = 3\angle 30^\circ$ and $z_2 = 3\angle -150^\circ$

\[
\begin{align*}
\text{a)} & \quad z_1 + z_2 = 2 + j3 + 1 - j2 = 3 + j = \sqrt{3^2 + 1^2} \angle \tan^{-1}\left(\frac{1}{3}\right) \approx 3.16 \angle 18.43^\circ \\
& \quad z_1 - z_2 = 2 + j3 - 1 + j2 = 1 + j5 = \sqrt{1^2 + 5^2} \angle \tan^{-1}\left(\frac{1}{5}\right) \approx 5.1 \angle 78.7^\circ \\
\text{b)} & \quad z_1 + z_2 = 3 - j3 = \sqrt{3^2 + (-3)^2} \angle \tan^{-1}\left(\frac{-3}{3}\right) \approx 4.24 \angle -45^\circ \\
& \quad z_1 - z_2 = 3 + j3 = \sqrt{3^2 + 3^2} \angle \tan^{-1}\left(\frac{3}{3}\right) \approx 4.24 \angle 45^\circ \\
\text{c)} & \quad z_1 + z_2 = 3\angle 30^\circ + 3\angle -30^\circ = 3 \cos 30^\circ + j3 \sin 30^\circ + 3 \cos(-30) + j3 \sin(-30) \\
& \approx 5.19 + j0 = \sqrt{5.19^2 + 0^2} \angle \tan^{-1}\left(\frac{0}{5.19}\right) \approx 5.19 \angle 0 \\
& \quad z_1 - z_2 = 3\angle 30^\circ - 3\angle -30^\circ = 3 \cos 30^\circ + j3 \sin 30^\circ - 3 \cos(-30) - j3 \sin(-30) \\
& \approx 0 + j3 = \sqrt{0^2 + 3^2} \angle \tan^{-1}\left(\frac{3}{0}\right) \approx 3 \angle 90^\circ \\
\text{d)} & \quad z_1 + z_2 = 3\angle 30^\circ + 3\angle -150^\circ = 3 \cos 30^\circ + j3 \sin 30^\circ + 3 \cos(-150) + j3 \sin(-150) \\
& \approx 0 + j0 = 0 \\
& \quad z_1 - z_2 = 3\angle 30^\circ - 3\angle -150^\circ = 3 \cos 30^\circ + j3 \sin 30^\circ - 3 \cos(-150) - j3 \sin(-150) \\
& \approx 5.19 + j3 = \sqrt{5.19^2 + 3^2} \angle \tan^{-1}\left(\frac{3}{5.19}\right) \approx 5.19 + j3 \\
\end{align*}
\]
1.25 A voltage source is given by \( v_s(t) = 25 \cos(2\pi \times 10^3 t - 30^\circ) \) is connected to a series RC load as shown in Fig. 1-20. If \( R = 1 \ \text{M}\Omega \) and \( C = 200\text{pF} \), obtain an expression for \( v_c(t) \), the voltage across the capacitor.

Essentially, this is a voltage divider, for which \( V_{\text{out}} = \frac{V_{\text{source}} \times R_{\text{partial}}}{R_{\text{total}}} \), where \( R_{\text{partial}} \) is the amount of resistance taking place after the point of the \( V_{\text{out}} \) measurement. However, since this uses a capacitor, one must use complex resistance values (also known as impedance), which will be noted here with a \( Z \).

\[ \omega = 2\pi \times 10^3 \]

From Table 1-5, \( \tilde{V}_s = 25e^{-j30} \)

\[
V_c = V_{\text{source}} \times \frac{Z_{\text{partial}}}{Z_{\text{total}}} \quad \Rightarrow \quad \tilde{V}_c = \frac{1}{\tilde{V}_s} \times \frac{1}{\frac{1}{R} + \frac{1}{j\omega C}} = \tilde{V}_s \times \frac{1}{j\omega RC + 1} \\
= \tilde{V}_s \times \frac{1}{\sqrt{(\omega RC)^2 + 1^2} e^{j\tan^{-1}(\omega RC)}} = \tilde{V}_s \times \frac{1}{\sqrt{(\omega RC)^2 + 1^2} e^{-j\tan^{-1}(\omega RC)}} \\
= \frac{25}{\sqrt{(\omega RC)^2 + 1^2} e^{-j(30+\tan^{-1}(\omega RC))}} \bigg|_{\omega RC=2\pi \times 10^3 \times 1 \times 10^6 \times 200 \times 10^{-12}=1.26} = 15.57e^{-j81.49} \quad \Rightarrow \\
\]

\[ v_c(t) = 15.57 \cos(2\pi \times 10^3 t - 81.5^\circ) \]

3.2 Given vectors \( \textbf{A} = \hat{x}2 - \hat{y}3 + \hat{z} \), \( \textbf{B} = \hat{x}2 - \hat{y} + \hat{z}3 \), and \( \textbf{C} = \hat{x}4 + \hat{y}2 - \hat{z}2 \), show that \( \textbf{C} \) is perpendicular to both \( \textbf{A} \) and \( \textbf{B} \).

Two vectors are perpendicular if their dot product is 0.

\[ \textbf{C} \cdot \textbf{A} = 2 \times 4 - 3 \times 2 - 1 \times 2 = 0 \]

\[ \textbf{C} \cdot \textbf{B} = 2 \times 4 - 1 \times 2 - 3 \times 2 = 0 \]
3.9 Find an expression for the unit vector directed toward the origin from an arbitrary point on the line described by \( x = 1 \) and \( z = -3 \).

The unit vector to be found is depicted, in red, by this figure:

At any point along this line, a vector pointing away from the origin is given by the coordinates of the point:

\[ 3\hat{x} + y\hat{y} - 3\hat{z}. \]

To reverse the direction, and thus point towards the origin, one needs only to take the negative of each vector component:

\[ -3\hat{x} - y\hat{y} + 3\hat{z}. \]

To make this a unit vector, divide by the magnitude:

\[
\frac{1}{\sqrt{3^2 + y^2 + (-3)^2}} (-3\hat{x} - y\hat{y} + 3\hat{z}) = \frac{1}{\sqrt{18 + y^2}} (-3\hat{x} - y\hat{y} + 3\hat{z})
\]
3.26 Find the volumes described by

a) \(2 \leq r \leq 5; \ \frac{\pi}{2} \leq \phi \leq \pi; 0 \leq z \leq 2\)

b) \(0 \leq R \leq 5; \ 0 \leq \theta \leq \frac{\pi}{3}; 0 \leq \phi \leq 2\pi\)

a) This can be solved by algebra. One need only to take the volume of a cylinder of radius 5 and height 2 and subtract from it the volume of a cylinder of radius 2 and height 2, which gives \((\pi 5^2 - \pi 2^2) \times 2 = 42\pi\). Additionally, a radial variance from \(\frac{\pi}{2} \leq \phi \leq \pi\) corresponds to only one quadrant, which means only one fourth of this volume should be counted. \(42\pi/4 = 21\pi/2\)

b) With density assumed to be 1, the integral in spherical coordinates takes the form:

\[
\begin{aligned}
&\int_0^{2\pi} \int_0^{\pi/3} \int_0^5 1 r^2 \sin \theta \ dr \ d\theta \ d\phi = \int_0^{2\pi} \int_0^{\pi/3} \int_3^{r=5} r^2 \sin \theta \ dr \ d\theta \ d\phi \\
&= \int_0^{2\pi} \int_0^{\pi/3} 125/3 \sin \theta \ d\theta \ d\phi = -125/3 \int_0^{2\pi} \cos \theta \bigg|_{\theta=\pi/3}^{\theta=\pi/2} \ d\phi = 125/6 \int_0^{2\pi} 1 \ d\phi \\
&\frac{125}{6} \bigg|_{\phi=0}^{\phi=2\pi} = \frac{125\pi}{3}
\end{aligned}
\]

3.33 Transform the vector \( \mathbf{A} = \hat{\mathbf{r}} \sin^2 \theta \cos \phi + \hat{\mathbf{\theta}} \cos^2 \phi - \hat{\phi} \sin \phi \) into cylindrical coordinates and then evaluate it at \( P = (2, \pi/2, \pi/2) \).

From table 3-2: \( A_r = A_R \sin \theta + A_\theta \cos \theta, A_\theta = A_\phi, A_z = A_R \cos \theta - A_\theta \sin \theta \) and \( R = \sqrt{r^2 + z^2}, \theta = \tan^{-1}(r/z), \phi = \phi \).

So, \( \mathbf{A} = \mathbf{f}(\sin^2 \theta \cos \phi \sin \theta + \cos^2 \phi \cos \theta) - \hat{\phi} \sin \phi + 2(\sin^2 \theta \cos \phi \cos \theta - \cos^2 \phi \sin \theta) \)

\( \mathbf{A} = \mathbf{f}(\sin^3 \theta \cos \phi + \cos^2 \phi \cos \theta) - \hat{\phi} \sin \phi + 2(\sin^2 \theta \cos \phi \cos \theta - \cos^2 \phi \sin \theta) \)

\( \mathbf{A} = \mathbf{f}(\sin^3 (\tan^{-1}(\phi/2))) \cos \phi + \cos^2 \phi \cos(\tan^{-1}(\phi/2))) - \hat{\phi} \sin \phi \\
+ 2(\sin^2 (\tan^{-1}(\phi/2))) \cos \phi \cos(\tan^{-1}(\phi/2))) - \cos^2 \phi \sin(\tan^{-1}(\phi/2)) \)

Since \( \phi = \pi/2 \) and \( \cos \pi/2 = 0 \), all terms cancel out except the \(-\hat{\phi} \sin \phi = -\hat{\phi} \)